

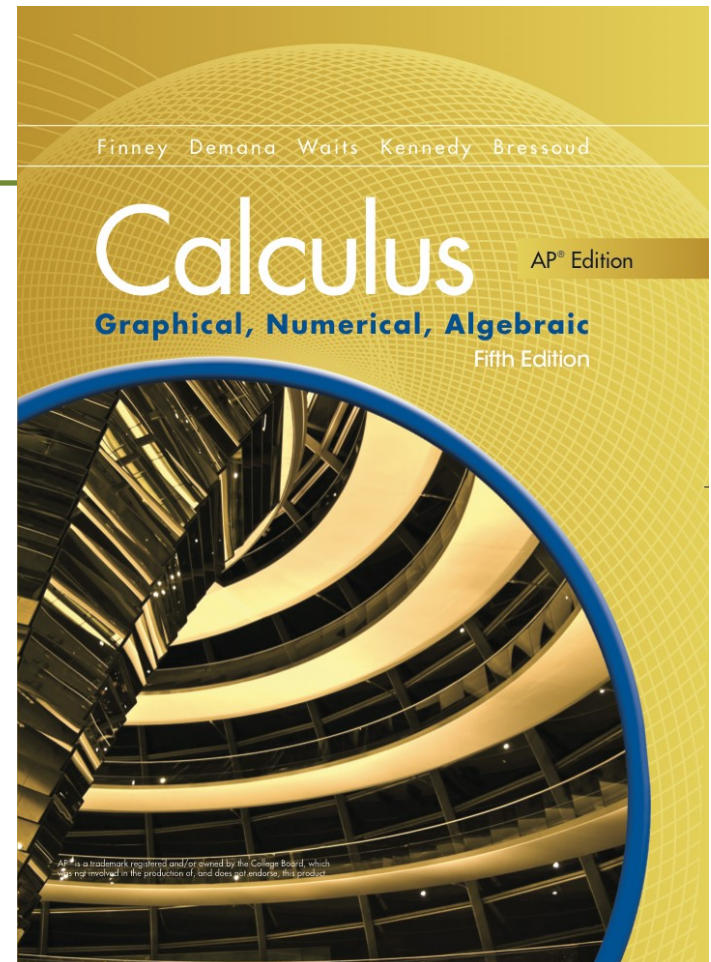
Chapter 2

Limits and Continuity



Section 2.4

Rates of Change and Tangent Lines



Quick Review

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B .

1. $A(-5, 2), B(3, 5)$

2. $A(1, 3), B(a, b)$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3), (5, -1)$

4. $(-3, -1), (3, 3)$

Quick Review

In Exercises 5 – 9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= \frac{3}{2}$

6. through $(1, 6)$ and $(4, -1)$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$

Quick Review

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$
9. through $(-1, 3)$ and parallel to $2x + 3y = 5$
10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $\frac{5}{3}$?

Quick Review Solutions

In Exercises 1 and 2, find the increments Δx and Δy from point A to point B .

1. $A(-5, 2), B(3, 5)$

$$\Delta x = 8, \quad \Delta y = 3$$

2. $A(1, 3), B(a, b)$

$$\Delta x = a - 1, \quad \Delta y = b - 3$$

In Exercises 3 and 4, find the slope of the line determined by the points.

3. $(-2, 3), (5, -1)$

$$-\frac{4}{7}$$

4. $(-3, -1), (3, 3)$

$$\frac{2}{3}$$

Quick Review Solutions

In Exercises 5 – 9, write an equation for the specified line.

5. through $(-2, 3)$ with slope $= \frac{3}{2}$ $y = \frac{3}{2}x + 6$

6. through $(1, 6)$ and $(4, -1)$ $y = -\frac{7}{3}x + \frac{25}{3}$

7. through $(1, 4)$ and parallel to $y = -\frac{3}{4}x + 2$ $y = -\frac{3}{4}x + \frac{19}{4}$

Quick Review Solutions

8. through $(1, 4)$ and perpendicular to $y = -\frac{3}{4}x + 2$

$$y = \frac{4}{3}x + \frac{8}{3}$$

9. through $(-1, 3)$ and parallel to $2x + 3y = 5$

$$y = -\frac{2}{3}x + \frac{7}{3}$$

10. For what value of b will the slope of the line through $(2, 3)$ and $(4, b)$ be $\frac{5}{3}$?

$$b = \frac{19}{3}$$

What you'll learn about

- Tangent lines
- Slopes of curves
- Instantaneous rate of change
- Sensitivity

...and why

The tangent line determines the direction of a body's motion at every point along its path

Average Rates of Change

The average rate of change of a quantity over a period of time is the amount of change divided by the time it takes.

In general, the *average rate of change* of a function over an interval is the amount of change divided by the length of the interval.

Also, the average rate of change can be thought of as the slope of a secant line to a curve.

Example Average Rates of Change

Find the average rate of change of $f(x) = 2x^2 - 3x + 7$
over the interval $[-2, 4]$

$$\begin{aligned}\frac{f(4) - f(-2)}{4 - (-2)} &= \frac{(2(-2)^2 - 3(-2) + 7) - (2(4)^2 - 3(4) + 7)}{4 - (-2)} \\ &= \frac{21 - 27}{-6} = \frac{-6}{-6} = 1\end{aligned}$$

Tangent to a Curve

In calculus, we often want to define the rate at which the value of a function $y = f(x)$ is changing with respect to x at any particular value $x = a$ to be the slope of the tangent to the curve $y = f(x)$ at $x = a$.

The problem with this is that we only have one point and our usual definition of slope requires two points.

Tangent to a Curve

The process becomes:

1. Start with what can be calculated, namely, the slope of a secant through P and a point Q nearby on the curve.
2. Find the limiting value of the secant slope (if it exists) as Q approaches P along the curve.
3. Define the *slope of the curve* at P to be this number and *define the tangent to the curve* at P to be the line through P with this slope.

Example Tangent to a Curve

Given $y = x^2 + 2$ at $x = -1$ find:

the slope of the curve and an equation of the tangent line.

Then draw a graph of the curve and tangent line in the same viewing window.

(a) Write an expression for the slope of the secant line and find the limiting value of the slope as Q approaches P along the curve.

When $x = -1$, $y = x^2 + 2 = 3$ so $P(-1, 3)$

$$\lim_{h \rightarrow 0} \frac{y(-1+h) - y(-1)}{h} = \lim_{h \rightarrow 0} \frac{(-1+h)^2 + 2 - [(-1)^2 + 2]}{h}$$

$$\lim_{h \rightarrow 0} \frac{3 - 2h + h^2 - 3}{h} = \lim_{h \rightarrow 0} \frac{h^2 - 2h}{h} = \lim_{h \rightarrow 0} \frac{h(h-2)}{h} = \lim_{h \rightarrow 0} (h-2) = -2$$

Example Tangent to a Curve

(b) The tangent line has slope - 2 and passes through $(-1, 3)$.

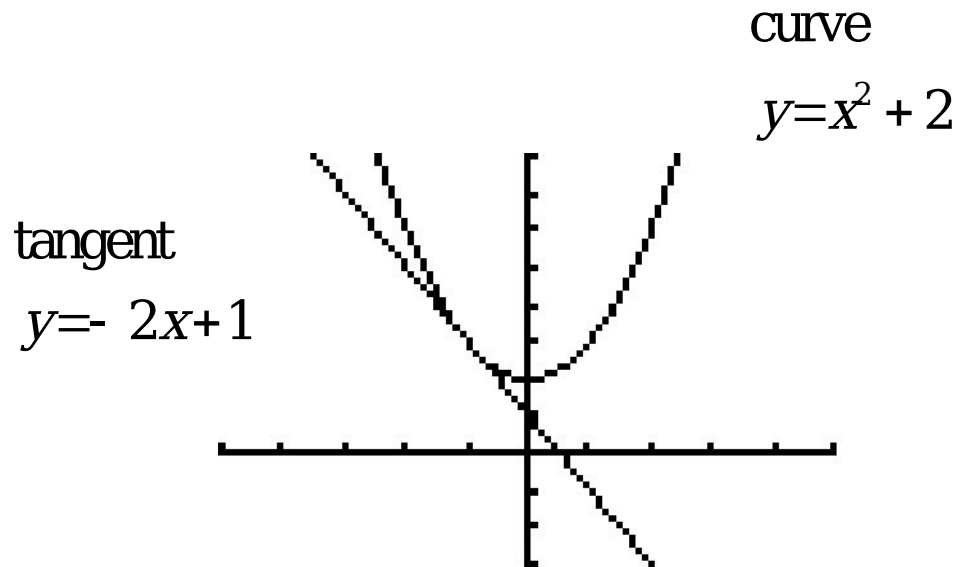
The equation of the tangent line is

$$y - 3 = -2(x - (-1))$$

$$y = -2(x + 1) + 3$$

$$y = -2x - 2 + 3$$

$$y = -2x + 1$$

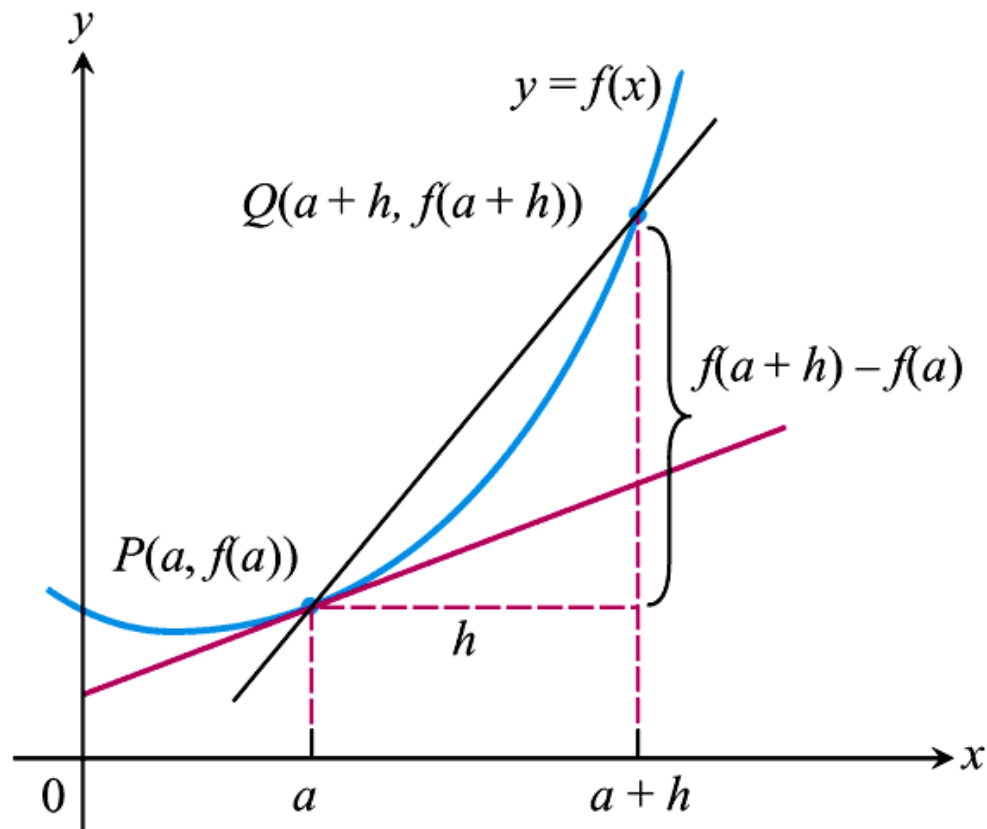


Slope of a Curve

To find the tangent to a curve $y = f(x)$ at a point $P(a, f(a))$ calculate the slope of the secant line through P and a point $Q(a+h, f(a+h))$. Next, investigate the limit of the slope as $h \rightarrow 0$.

If the limit exists, it is the slope of the curve at P and we define the tangent at P to be the line through P with this slope

Slope of a Curve



Slope of a Curve at a Point

The **slope of the curve** $y = f(x)$ at the point $P(a, f(a))$ is the number

$$m = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

provided the limit exists.

The **tangent line to the curve** at P is the line through P with this slope.

Slope of a Curve

All of the following mean the same:

1. the slope of $y = f(x)$ at $x = a$
2. the slope of the tangent to $y = f(x)$ at $x = a$
3. the (instantaneous) rate of change of $f(x)$ with respect to x at $x = a$
4. $\lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$

The expression $\frac{f(a+h) - f(a)}{h}$ is the **difference quotient** of f at a .

Normal to a Curve

The normal line to a curve at a point is the line perpendicular to the tangent at the point.

The slope of the normal line is the negative reciprocal of the slope of the tangent line.

Example Normal to a Curve

Given $y = x^2 + 2$ at $x = -1$ write the equation of the normal line.

Draw a graph of the curve, the tangent line and the normal line in the same viewing window.

From an earlier example, the slope of the tangent line was found

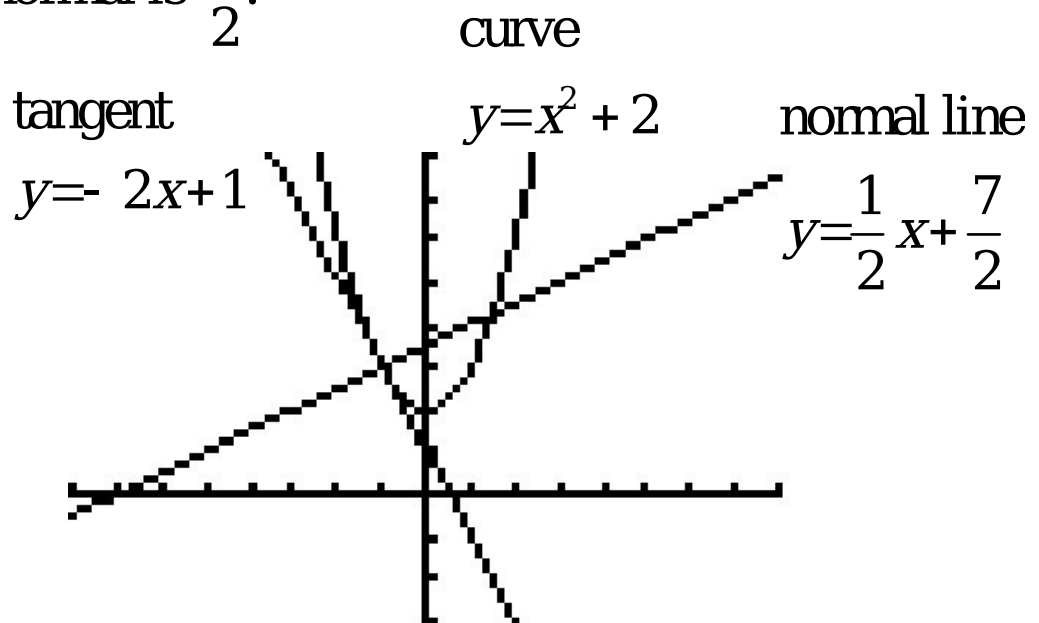
to be -2 so the slope of the normal is $\frac{1}{2}$.

$$y - 3 = \frac{1}{2}(x - (-1))$$

$$y = \frac{1}{2}(x + 1) + 3$$

$$y = \frac{1}{2}x + \frac{1}{2} + \frac{6}{2}$$

$$y = \frac{1}{2}x + \frac{7}{2}$$



Speed Revisited

The function $y=16t^2$ is an object's **position function**. An object's average speed along a coordinate axis for a given period of time is the average rate of change of its position $y = f(t)$.

Its **instantaneous speed** at any time t is the **instantaneous rate of change** of position with respect to time at time t , or $\lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$.

Sensitivity

We live in an interconnected world where changes in one quantity cause changes in another. For example, crop yields per acre depend on rainfall. If rainfall has been low, each small increase in the amount of rain creates a small increase in crop yield. For a drug that works to lower a patient's temperature, each small increase in the amount of the drug will lower the temperature a small amount.

Sensitivity

The mathematical connection between such changes is known as **sensitivity**. Sensitivity describes how one variable responds to small changes in another variable.

If we let T denote the patient's temperature and D the dosage, then the sensitivity is given by

$$\text{sensitivity} = \lim_{\Delta D \rightarrow 0} \frac{\Delta T}{\Delta D}$$

Quick Quiz Sections 2.3 and 2.4

You may use a graphing calculator to solve the following problems.

1. Which of the following values is the average rate of change

of $f(x) = \sqrt{x+1}$ over the interval $[0, 3]$?

(A) - 3

(B) - 1

(C) $-\frac{1}{3}$

(D) $\frac{1}{3}$

(E) 3

Quick Quiz Sections 2.3 and 2.4

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1. Which of the following values is the average rate of change of $f(x) = \sqrt{x+1}$ over the interval $[0, 3]$?

(A) -3

(B) -1

(C) $-\frac{1}{3}$

(D) $\frac{1}{3}$

(E) 3

Quick Quiz Sections 2.3 and 2.4

2. Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \leq x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \leq 6 \\ 1, & 6 < x < 8 \end{cases} ?$$

- (A) $\lim_{x \rightarrow 4} f(x)$ exists
- (B) $f(4)$ exists
- (C) $\lim_{x \rightarrow 6} f(x)$ exists
- (D) $\lim_{x \rightarrow 8} f(x)$ exists
- (E) f is continuous at $x=4$

Quick Quiz Sections 2.3 and 2.4

2. Which of the following statements is false for the function

$$f(x) = \begin{cases} \frac{3}{4}x, & 0 \leq x < 4 \\ 2, & x = 4 \\ -x + 7, & 4 < x \leq 6 \\ 1, & 6 < x < 8 \end{cases} ?$$

(A) $\lim_{x \rightarrow 4} f(x)$ exists

(B) $f(4)$ exists

(C) $\lim_{x \rightarrow 6} f(x)$ exists

(D) $\lim_{x \rightarrow 8} f(x)$ exists

(E) f is continuous at $x=4$

Quick Quiz Sections 2.3 and 2.4

3. Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

(A) $y = \frac{1}{4}x + \frac{9}{2}$

(B) $y = -4x + 13$

(C) $y = -4x - 3$

(D) $y = 4x - 3$

(E) $y = 4x + 13$

Quick Quiz Sections 2.3 and 2.4

3. Which of the following is an equation for the tangent line to $f(x) = 9 - x^2$ at $x = 2$?

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(C) $y = -4x - 3$

(D) $y = 4x - 3$

(E) $y = 4x + 13$

Chapter Test

In Exercises 1 and 2, find the limits.

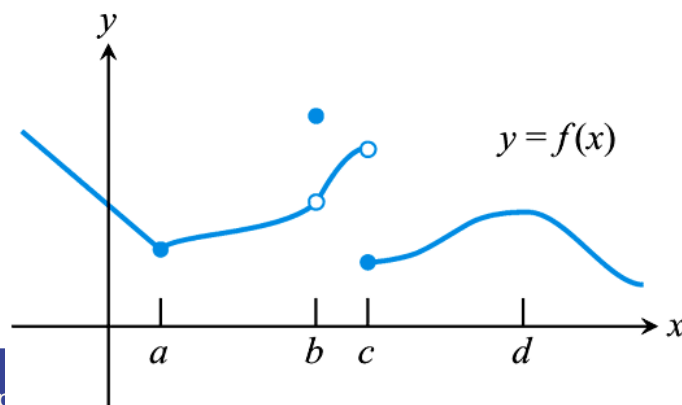
$$1. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x}$$

$$2. \lim_{x \rightarrow \infty} \frac{x + \sin x}{x + \cos x}$$

In Exercises 3 and 4, determine whether the limit exists on the basis of the graph of $y = f(x)$. The domain of f is the set of real numbers.

$$3. \lim_{x \rightarrow c^-} f(x)$$

$$4. \lim_{x \rightarrow b} f(x)$$



Chapter Test

In Exercise 5, use the graph of the function with domain $-1 \leq x \leq 3$.

5. Determine

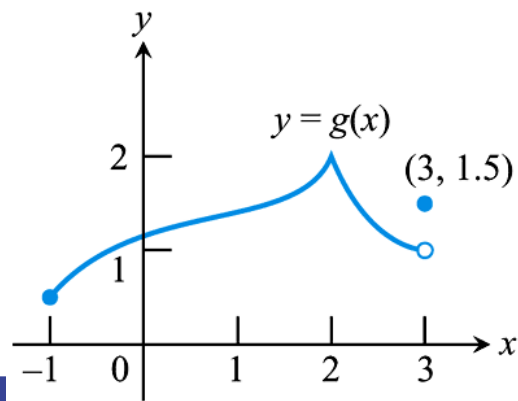
(a) $\lim_{x \rightarrow 3} g(x)$

(b) $g(3)$.

(c) whether $g(x)$ is continuous at $x=3$.

(d) the points of discontinuity of $g(x)$.

(e) *Writing to Learn* whether any points of discontinuity are removable. If so, describe the extended function. If not, explain why not.



Chapter Test

In Exercise 6, (a) find the vertical asymptotes of the graph of $y = f(x)$, and (b) describe the behavior of $f(x)$ to the right and left of any vertical asymptote.

6.
$$f(x) = \frac{x-1}{x^2(x+2)}$$

Chapter Test

7. Given

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

- (a) Find the right-hand and left-hand limits of f at $x = -1, 0$ and 1 .
- (b) Does f have a limit as x approaches $-1? 0? 1?$
If so, what is it? If not, why not?
- (c) Is f continuous at $x = -1? 0? 1?$ Explain.

Chapter Test

In Exercise 8, find (a) a power function end behavior model and
(b) any horizontal asymptotes.

8. $f(x) = \frac{2x+1}{x^2 - 2x+1}$

9. Find the average rate of change of $f(x) = 1 + \sin x$
over the interval $[0, \frac{\pi}{2}]$.

Chapter Test

10. Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$.

Find (a) the slope of the curve $y = f(x)$ at P , (b) an equation of the tangent at P and (c) an equation of the normal at P .

Chapter Test Solutions

In Exercises 1 and 2, find the limits.

$$1. \lim_{x \rightarrow 0} \frac{\frac{1}{2+x} - \frac{1}{2}}{x} = -\frac{1}{4} \qquad 2. \lim_{x \rightarrow \infty} \frac{x + \sin x}{x + \cos x} = 1$$

In Exercises 3 and 4, determine whether the limit exists on the basis of the graph of $y = f(x)$. The domain of f is the set of real numbers.

$$3. \lim_{x \rightarrow c} f(x) \text{ Exists} \qquad 4. \lim_{x \rightarrow b} f(x) \text{ Exists}$$

Chapter Test Solutions

In Exercise 5, use the graph of the function with domain - $1 \leq x \leq 3$.

5. Determine

- (a) $\lim_{x \rightarrow 3} g(x)$ 1 (b) $g(3)$. 1.5
- (c) whether $g(x)$ is continuous at $x=3$. No
- (d) the points of discontinuity of $g(x)$. at $x=3$ and points
not in the domain
- (e) *Writing to Learn* whether any points of discontinuity
are removable. If so, describe the extended function. If not,
explain why not.
Removable at $x=3$ by assigning the value 1 to $g(3)$.

Chapter Test Solutions

In Exercise 6, (a) find the vertical asymptotes of the graph of $y = f(x)$, and (b) describe the behavior of $f(x)$ to the right and left of any vertical asymptote.

6. $f(x) = \frac{x-1}{x^2(x+2)}$ vertical asymptotes: $x=0$, $x=-2$

At $x=0$

$$\text{Left-hand limit} = \lim_{x \rightarrow 0^-} \frac{x-1}{x^2(x+2)} = -\infty \quad \text{Right-hand limit} = \lim_{x \rightarrow 0^+} \frac{x-1}{x^2(x+2)} = -\infty$$

At $x=-2$

$$\text{Left-hand limit} = \lim_{x \rightarrow -2^-} \frac{x-1}{x^2(x+2)} = \infty \quad \text{Right-hand limit} = \lim_{x \rightarrow -2^+} \frac{x-1}{x^2(x+2)} = -\infty$$

Chapter Test Solutions

7. Given

$$f(x) = \begin{cases} 1, & x \leq -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

- (a) Find the right-hand and left-hand limits of f at $x = -1, 0$ and 1 .
- (b) Does f have a limit as x approaches -1 ? 0 ? 1 ?
If so, what is it? If not, why not?
- (c) Is f continuous at $x = -1$? 0 ? 1 ? Explain.

Chapter Test Solutions

7.(a) At $x = -1$:

$$\text{Left-hand limit} = \lim_{x \rightarrow -1^-} f(x) = \lim_{x \rightarrow -1^-} (1) = 1$$

$$\text{Right-hand limit} = \lim_{x \rightarrow -1^+} f(x) = \lim_{x \rightarrow -1^+} (-x) = 1$$

At $x = 0$:

$$\text{Left-hand limit} = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x) = 0$$

$$\text{Right-hand limit} = \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (-x) = 0$$

At $x = 1$

$$\text{Left-hand limit} = \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (-x) = -1$$

$$\text{Right-hand limit} = \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1) = 1$$

Chapter Test Solutions

- 7.(b) At $x = -1$: Yes, the limit is 1.
At $x = 0$: Yes, the limit is 0.
At $x = 1$: No, the limit doesn't exist because the two one-sided limits are different.
- (c) At $x = -1$: Continuous because $f(-1) = \text{the limit}$.
At $x = 0$: Discontinuous because $f(0) \neq \text{the limit}$.
At $x = 1$: Discontinuous because the limit does not exist.

Chapter Test Solutions

In Exercise 8, find (a) a power function end behavior model and
(b) any horizontal asymptotes.

8. $f(x) = \frac{2x+1}{x^2 - 2x+1}$ (a) $\frac{2}{x}$ (b) $y=0$

9. Find the average rate of change of $f(x) = 1 + \sin x$
over the interval $\left[0, \frac{\pi}{2}\right]$. $\frac{2}{\pi}$

Chapter Test Solutions

10. Let $f(x) = x^2 - 3x$ and $P = (1, f(1))$.

Find (a) the slope of the curve $y = f(x)$ at P , (b) an equation of the tangent at P and (c) an equation of the normal at P .

(a) $m = -1$ (b) $y = -x - 1$ (c) $y = x - 3$